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Stress Potentials on C^1 Domains

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We obtain the stress function from the boundary stress via a device which we call the modified lower order potential. The boundary value problem considered here is a fundamental problem in the theory of elasticity in two dimensions. The results obtained here give point-wise solutions to a Neumann-type boundary value problem for the biharmonic equation and extend previous results to include domains with C^1 boundaries and with boundary data in $L^q \times L^q(\partial\Omega)$.

The modified lower order potential was introduced by Cohen and Gosselin in [4] to solve the problem studied here. In [4] the solution was obtained with boundary data in the space of cosets of linear functionals acting on the space of Dirichlet data. The convergence at the boundary was devised to fit the boundary data and was somewhat awkward. In this paper we show that the solution can be obtained with boundary data in a subspace of $L^q \times L^q(\partial\Omega)$ and with the boundary values obtained nontangentially point-wise almost everywhere.

The problem studied here is equivalent to obtaining the interior stress on a thin elastic plate from the stress on the boundary. The interior stresses are given by the stress potential, a vector valued potential obtained as a pair of differential operators acting on the modified lower order potential. The form of the stress potential makes the analysis at the boundary obtainable via the singular integral estimates of Calderón [2].

In Section 2.3 we prove an extension of Theorem 2 of Calderón's paper on the Cauchy integral on Lipschitz curves [2] to a wider class of singular integrals. This result, which is essentially an observation about Calderón's proof, is necessary to obtain the convergence of certain potentials at the boundary.

The problem in elastostatics of obtaining the interior stress from the stress on the boundary can be treated as a boundary value problem for a second order elliptic system satisfied by the components of displacement. Recently, Dahlberg, Kenig, and Verchota [8], using the system of layer potentials in [9], obtained a solution for this system with L^2 boundary data on Lipschitz domains. Their result extends to L^p data for $1<p<2$ but fails on Lipschitz domains for the range $2<p<\infty$.

In this paper we obtain a solution to a biharmonic problem on C^1 domains with data in L^p for $1<p<\infty$. The extra smoothness in the C^1 case allows the problem to be treated using compactness arguments and Fredholm theory. It is interesting to note that the layer potentials utilized by Dahlberg *et al.* do not give the full C^1 results via Fredholm theory since some of the boundary operators fail to be compact.

1. THE MAIN THEOREM

In [4], Cohen and Gosselin introduced the modified multiple layer potential u_m, a new representation of Agmon's multiple layer potential (see Sect. 6 of [1]) for the biharmonic equation. The modified multiple layer potential is obtained from Agmon's by applying a biharmonic analog of the Cauchy Riemann equations to the kernels of the multiple layer potential. It is easily shown to satisfy a version of the Dirichlet problem where a biharmonic function is sought with gradient in $L^p \times L^p(\partial\Omega)$ (see Sect. 3 of [4]).

It is shown that the interior nontangential limit of ∇u_m is of the form $(\bar{I}+\bar{\mathscr{L}})\,\bar{g}$ where $\bar{g}$ is the density of the potential. The adjoint $(\bar{I}+\bar{\mathscr{L}})^*$ is shown to be the exterior "weak" limit of a vector of second order differential operators acting on a biharmonic function which we call the modified lower order potential.

In what follows we will assume that Ω is a bounded simply connected C^1 domain in $\mathbb{R}^2$. The letter X will usually denote a point in Ω or in $\bar{\Omega}^c$. The letters P and Q will usually denote points on the boundary $\partial\Omega$.

1.1. *The Modified Potentials*

We begin with some preliminaries.

Definition (1.1.1). Let $\mathscr{C}_p=\{\bar{g}=(g,h)\in L^p(\partial\Omega)\times L^p(\partial\Omega)\colon \int_{\partial\Omega} g\,dx + h\,dy=0\}$.

DEFINITION (1.1.2). The modified multiple layer potential $u_m=u_m(\bar{g};X)$ is defined by

$$u_m(\bar{g};X)=2\int_{\partial\Omega_0} g(Q)\,L_1^Q\tilde{F}(X-Q)+h(Q)\,L_2^Q\tilde{F}(X-Q)\,ds(Q) \qquad (1.1.3)$$

where for $X=(x, y)$, $\tilde{F}(X)=(-1/4\pi)\{x^2+y^2)\arg(x+iy)-xy\}=(-1/4\pi)$ $\operatorname{Im}\{\bar{z}z\log z-\frac{1}{2}\bar{z}z+\frac{1}{2}z^2\}$. For $Q\in\partial\Omega$,

$$L_1^Q=x_s(Q)\left(\frac{\partial^2}{\partial x^2}\right)^Q+y_s(Q)\left(\frac{\partial^2}{\partial x\partial y}\right)^Q,$$

$$L_2^Q=x_s(Q)\left(\frac{\partial^2}{\partial x\partial y}\right)^Q+y_s(Q)\left(\frac{\partial^2}{\partial y^2}\right)^Q,$$

the superscript Q denotes the point at which the differential operator is acting and $x_s(Q)\,\mathbf{i}+y_s(Q)\,\mathbf{j}$ is the unit tangent in the counterclockwise direction. The integration begins at a point P_0 and proceeds counterclockwise around $\partial\Omega$. The designation of a point P_0 enables us to define a branch of the argument which appears in the formula for $\tilde{F}$.

We then get an integral representation:

$$\nabla u_m(X)=\int_{\partial\Omega}\bar{g}(Q)\,l(X, Q)\,ds(Q), \tag{1.1.4}$$

where

$$l(X, Q)=2\begin{bmatrix} L_1^Q\,\partial_x^X\tilde{F}(X-Q) & L_1^Q\,\partial_y^X\tilde{F}(X-Q) \\ L_2^Q\,\partial_x^X\tilde{F}(X-Q) & L_2^Q\,\partial_y^X\tilde{F}(X-Q)\end{bmatrix}. \tag{1.1.5}$$

For $P\neq Q$ and $P\in\partial\Omega$ we can replace X in (1.1.5) with P and the matrix is still defined. We tentatively define $\mathscr{L}\bar{g}(P)=\text{p.v.}\int_{\partial\Omega}\bar{g}(Q)\;l(P, Q)\,ds(Q)$. We also tentatively define the formal adjoint,

$$\mathscr{L}^*\phi(Q)=\text{p.v.}\int_{\partial\Omega}l(P, Q)\,\boldsymbol{\phi}(P)^{\mathrm{T}}\,ds(P) \tag{1.1.6}$$

where $\boldsymbol{\phi}=(\phi, \psi)\in L^q(\partial\Omega)\times L^q(\partial\Omega)$ and the supercript T indicates that we are looking at the transpose of the row vector (ϕ, ψ). If we formally interchange differentiation and integration we have

$$\mathscr{L}^*\phi(Q)=(L_1^Q, L_2^Q)^{\mathrm{T}}\,2\int_{\partial\Omega}\partial_x^P\tilde{F}(P-Q)\,\phi(P)+\partial_y^P\tilde{F}(P-Q)\,\psi(P)\,ds(P). \tag{1.1.7}$$

This calculation suggests that the adjoint operator is obtained by applying the differential operator (L_1, L_2) to the integral potential on the right hand side of (1.1.7). To make this precise we must extend the right-hand side of (1.1.7) from points $Q\in\partial\Omega$ to points $X\in\Omega$.

DEFINITION (1.1.8). For $\boldsymbol{\phi} = (\phi, \psi) \in L^q(\partial\Omega) \times L^q(\partial\Omega)$ we define the modified lower order potential with density $\boldsymbol{\phi}$ by

$$v_m(\boldsymbol{\phi}; X) = 2 \int_{\partial\Omega_0} \partial_x^P \tilde{F}(P-X)\, \phi(P) + \partial_y^P \tilde{F}(P-X)\, \psi(P)\, ds(P), \quad (1.1.9)$$

where the function $\tilde{F}$ and the path of integration are as in the definition of the modified multiple layer potential (see definition (1.1.2)).

In what follows $x(s)\,\mathbf{i} + y(s)\,\mathbf{j}$ will denote the arclength parametrization of $\partial\Omega$ and $x_s(Q)\,\mathbf{i} + y_s(Q)\,\mathbf{j}$ is the unit tangent to $\partial\Omega$ at Q where $x_s = dx/ds$ and $y_s = dy/ds$.

DEFINITION (1.1.10)). For $Q \in \partial\Omega$, $X \in \Omega$, $\mathbf{n}_Q = y_s(Q)\,\mathbf{i} - x_s(Q)\,\mathbf{j}$, the unit outer normal to $\partial\Omega$ at Q and $V \in C^2(\Omega)$ we define

$$\mathbf{L}_{(\mathbf{n}_Q)} V(X) = (L^1_{(\mathbf{n}_Q)}\mathbf{i} + L^2_{(\mathbf{n}_Q)}\mathbf{j})\, V(X),$$

where

$$L^1_{(\mathbf{n}_Q)} V(X) = V_{xx}(X)\, x_s(Q) + V_{xy}(X)\, y_s(Q),$$
$$L^2_{(\mathbf{n}_Q)} V(X) = V_{xy}(X)\, x_s(Q) + V_{yy}(X)\, y_s(Q).$$

DEFINITION (1.1.11). For $v_m(\boldsymbol{\phi}; X)$ the modified lower order potential with density $\boldsymbol{\phi}$, $Q \in \partial\Omega$ and $\mathbf{L}_{(\mathbf{n}_Q)}$ the differential operator defined by (1.1.10), the stress potential is the vector $\mathscr{L}^*_{\boldsymbol{\phi}}(X)$ given by

$$\begin{aligned} \mathscr{L}^*_{\boldsymbol{\phi}}(X) &= \mathbf{L}_{(\mathbf{n}_Q)} v_m(\boldsymbol{\phi}; X) \\ &= \int_{\partial\Omega} l^*(P, X; \mathbf{n}_Q)\, \boldsymbol{\phi}(P)^T\, ds(P), \end{aligned} \quad (1.1.12)$$

where

$$l^*(P, X; \mathbf{n}_Q) = 2 \begin{bmatrix} L^1_{(\mathbf{n}_Q)}\, \partial_x^P \tilde{F}(P-X) & L^1_{(\mathbf{n}_Q)}\, \partial_y^P \tilde{F}(P-X) \\ L^2_{(\mathbf{n}_Q)}\, \partial_x^P \tilde{F}(P-X) & L^2_{(\mathbf{n}_Q)}\, \partial_y^P \tilde{F}(P-X) \end{bmatrix} \quad (1.1.13)$$

and the superscripts in the matrix indicate the point at which the differential operators are applied.

It is important to note that if we let $X = Q$ we have $l^*(P, Q; \mathbf{n}_Q) = l(P, Q)$, the matrix in formula (1.1.5).

1.2. *The Main Result*

In the paper, "Adjoint boundary value problems for the biharmonic equation on C^1 domains in the plane," Cohen and Gosselin [4] obtained a distributional solution of the problem $\Delta^2 V = 0$ in Ω, $\mathbf{L}_{(\mathbf{n}_Q)} V = \bar{\phi}$ on $\partial\Omega$. In that paper, $\bar{\phi}$ is in a subset of the dual of $\mathscr{C}_p$ (see (1.1.1) for the definition of $\mathscr{C}_p$) a coset of pairs of $L^q(\partial\Omega)$ functions satisfying certain moment conditions, and the convergence of $\mathbf{L}_{(\mathbf{n}_Q)}V$ is as linear functionals acting on elements in $\mathscr{C}_p$.

This paper shows that the convergence of $\mathbf{L}_{(\mathbf{n}_Q)}V$ actually occurs non-tangentially point-wise almost everywhere. Furthermore, the restriction of the boundary data to cosets is unnecessary and we can solve the boundary value problem for pairs of $L^q(\partial\Omega)$ functions satisfying the appropriate moment conditions.

For $Q \in \partial\Omega$, let $\mathbf{N}_Q$ be the unit *inner* normal at Q. Let $\Gamma = \Gamma_\alpha^\delta(Q) = \{X \in \Omega;\ \langle X - Q, \mathbf{N}_Q \rangle > \alpha |X - Q|,\ |X - Q| < \delta,\ 0 < \alpha < 1\}$. We can now state the main theorem.

THEOREM (1.2.1). *If Ω is a simply connected, bounded C^1 domain in $\mathbb{R}^2$, $\boldsymbol{\phi} \in L^q(\partial\Omega) \times L^q(\partial\Omega)$ and satisfies the moment conditions:*

$$\int_{\partial\Omega} \phi\, ds = \int_{\partial\Omega} \psi\, ds = 0, \tag{1.2.2}$$

$$\int_{\partial\Omega} x\phi + y\psi\, ds = 0, \tag{1.2.4}$$

then there exists a function V, possibly multiple valued satisfying

$$\begin{aligned} &\Delta^2 V(X) = 0 && \text{for} \quad X \in \Omega \\ &\lim_{X \to Q} \mathbf{L}_{(\mathbf{n}_Q)} V(X) = \phi(Q)\,\mathbf{i} + \psi(Q)\,\mathbf{j} && \text{a.e. for} \quad X \in \Gamma(Q) \cap \Omega. \end{aligned} \tag{1.2.4}$$

Furthermore, there exists a pair $\boldsymbol{\phi}_1 = (\phi_1, \psi_1)$ such that the solution is given by the modified lower order potential $v_m(\boldsymbol{\phi}_1; X)$.

1.3. *The Connection with Elasticity*

The differential operator $\mathbf{L}$ defined in (1.1.10) arose in this paper from considering the adjoint of the trace of ∇u_m on the boundary. It is important to note that it also arises very naturally from the theory of elasticity in two dimensions.

At a point P in a thin plate we let $X_n\mathbf{i} + Y_n\mathbf{j} = \mathscr{S}(\mathbf{n}_p)$ denote the force per unit length exerted along a small line segment dl, which passes through the point P and which is perpendicular to the unit vector $\mathbf{n}_p$. For $\mathbf{i}_p$, $\mathbf{j}_p$ the usual unit vectors in the x and y directions at P we define $\mathscr{S}(\mathbf{i}_P) = (X_x, Y_x)$, $\mathscr{S}(\mathbf{j}_P) = (X_y, Y_y)$.

It follows from equations going back to Cauchy, (see [10]), that for $\mathbf{n}_P = n_1\mathbf{i} + n_2\mathbf{j}$, the force $\mathscr{S}(\mathbf{n}_P)$ is related to the forces $\mathscr{S}(\mathbf{i}_P)$ and $\mathscr{S}(\mathbf{j}_P)$ by the equation $\left[\begin{smallmatrix} X_n \\ Y_n \end{smallmatrix}\right] = \left[\begin{smallmatrix} X_x & X_y \\ Y_x & Y_y \end{smallmatrix}\right] \left[\begin{smallmatrix} n_1 \\ n_2 \end{smallmatrix}\right]$.

Under the assumption of no "body" forces present the components of the above matrix satisfy the equilibrium equations

$$\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}=0,$$
$$\frac{\partial X_y}{\partial x}+\frac{\partial Y_y}{\partial y}=0, \tag{1.3.1}$$

and furthermore

$$\Delta(X_x+Y_y)=0. \tag{1.3.2}$$

Assuming that the matrix $\left[\begin{smallmatrix} X_x & Y_x \\ X_y & Y_y \end{smallmatrix}\right]$ is symmetric ($X_y=Y_x$), (1.3.1) implies there exists a function V such that $V_{xx}=Y_y$, $V_{xy}=-X_y=-Y_x$ and $V_{yy}=X_x$. Applying (1.3.2) we get $\Delta^2 V=0$.

For $\boldsymbol{\omega}=\beta\mathbf{i}-\alpha\mathbf{j}$ we let $\mathbf{L}_{(\boldsymbol{\omega})}V(X) = (\alpha V_{xx}(X)+\beta V_{xy}(X))\,\mathbf{i}+(\alpha V_{xy}(X)+\beta V_{yy}(X))\,\mathbf{j}$. Assuming that V is the potential for the equilibrium equations (1.3.1) and (1.3.2) we have for $\boldsymbol{\omega}=y_s(Q)\,\mathbf{i}-x_s(Q)\,\mathbf{j}$, that at point X

$$\begin{aligned}\mathscr{S}(\boldsymbol{\omega}_X)&=\begin{bmatrix} V_{yy} & -V_{xy} \\ -V_{xy} & V_{xy}\end{bmatrix}\begin{bmatrix} y_s \\ -x_s\end{bmatrix}\\ &=\begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}\begin{bmatrix} V_{xx}(X)\,x_s(Q)+V_{xy}(X)\,y_s(Q) \\ V_{xy}(X)\,x_s(Q)+V_{yy}(X)\,y_s(Q)\end{bmatrix}\\ &=E\mathbf{L}_{(\mathbf{n}_Q)}V(X),\end{aligned} \tag{1.3.3}$$

where $E=\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right]$.

From Theorem (1.2.1) we can solve $\Delta^2 V=0$, $\mathbf{L}_{(\mathbf{n}_Q)}V=(\phi,\psi)\in(L^q\times L^q)_0$. Hence for $\phi=-Y_n$, $\psi=X_n$ satisfying the moment conditions

$$\int_{\partial\Omega} Y_n(Q)\,ds(Q)=\int_{\partial\Omega} X_n(Q)\,ds(Q)=0 \tag{1.3.4}$$

and

$$\int_{\partial\Omega} y(Q)\,X_n(Q)-x(Q)\,Y_n(Q)\,ds(Q)=0$$

we can find a biharmonic function V such that $\mathbf{L}_{(\mathbf{n}_Q)}V(X)\to(-Y_n(Q), X_n(Q))$ as $X\to Q\in\partial\Omega$. By (1.3.3) we have $\mathscr{S}(\boldsymbol{\omega}_X)\to(X_n, Y_n)$ as $X\to\partial\Omega$. Thus we have found the interior stress from the boundary stress, for bounded simply connected domains with C^1 boundaries where the boundary stress (X_n, Y_n) are pairs of L^q functions satisfying the moment conditions in (1.3.4).

C

2. THE STRESS KERNELS

In Section 1 we introduced the stress potential $\mathscr{L}^*_\phi(X)=\int_{\partial\Omega} l^*(P, X; \mathbf{n}_Q)$ $\boldsymbol{\phi}(P)\ ds(P)$ where $l^*(P, X; \mathbf{n}_Q)$ is the 2×2 matrix in (1.1.12). We let $l^*_{ij}(P, X; \mathbf{n}_Q)$ denote the components of the matrix l^* and note that when $X=Q\in\partial\Omega$, $(Q\neq P)$, $l^*(P, Q; \mathbf{n}_Q)=l(P, Q)$ where $l(P, Q)$ is given in (1.1.5).

In this section we will analyze the integral operators whose kernels are the components l^*_{ij}. We will compute explicit formulae for the l^*_{ij}'s in rectangular coordinates and show how they are transformed under a rotation of coordinates. While the kernels are not invariant under rotations they can be analyzed using estimates of Calderón [2] for the Cauchy integral along Lipschitz curves.

2.1. *The Components of the Stress Matrix*

For $X=(x, y)$ and $\tilde{F}(X)=(-1/4\pi)\{(x^2+y^2)\arg(x+iy)-xy\}$ (where the argument is suitably chosen), we have by direct computation:

$$\begin{aligned}
\tilde{F}_{xxx}(X)&=\frac{2}{\pi}\frac{y^3}{(x^2+y^2)^2},\\
\tilde{F}_{xxy}(X)&=\frac{-2}{\pi}\frac{xy^2}{(x^2+y^2)^2},\\
\tilde{F}_{xyy}(X)&=\frac{2}{\pi}\frac{x^2y}{(x^2+y^2)^2},\\
\tilde{F}_{yyy}(X)&=\frac{-2}{\pi}\frac{x^3}{(x^2+y^2)^2}.
\end{aligned}\tag{2.1.1}$$

To compute the components of the stress matrix we let $X=(x, y)$, $P=(u, v)$ be the Cartesian coordinates for X and P and write $\mathbf{n}_Q=\beta\mathbf{i}-\alpha\mathbf{j}$. Then

$$\begin{aligned}
l_{11}(P, X; \mathbf{n}_Q)&=2\tilde{F}_{xxx}(P-X)\,x_s(Q)+2\tilde{F}_{xxy}(P-X)\,y_s(Q)\\
&=\frac{2}{\pi}\frac{\alpha(v-y)^3}{((x-u)^2+(v-y)^2)^2}-\frac{2}{\pi}\frac{\beta(u-x)(v-y)^2}{((u-x)^2+(v-y)^2)^2}\\
l_{12}(P, X; \mathbf{n}_Q)&=2\tilde{F}_{xxy}(P-X)\,x_s(Q)+2\tilde{F}_{xyy}(P-X)\,y_s(Q)\\
&=-\frac{2}{\pi}\frac{\alpha(u-x)(v-y)^2}{((u-x)^2+(v-y)^2)^2}+\frac{2}{\pi}\frac{\beta(u-x)^2(v-y)}{((u-x)^2+(v-y)^2)^2}\\
l_{21}(P, X; \mathbf{n}_Q)&=l_{12}(P, X; \mathbf{n}_Q)\\
l_{22}(P, X; \mathbf{n}_Q)&=2\tilde{F}_{xyy}(P-X)\,x_s(Q)+2\tilde{F}_{yyy}(P-X)\,y_s(Q)\\
&=\frac{2}{\pi}\frac{\alpha(u-x)^2(v-y)}{((u-x)^2+(v-y)^2)^2}-\frac{2}{\pi}\frac{\beta(u-x)^3}{((u-x)^2+(v-y)^2)^2}.
\end{aligned}\tag{2.1.2}$$

We next set,

$$
\begin{aligned}
J_0(x) &= \frac{2}{\pi}\frac{1}{(1+x^2)^2},\\
J_1(x) &= \frac{-2}{\pi}\frac{x}{(1+x^2)^2},\\
J_2(x) &= \frac{2}{\pi}\frac{x^2}{(1+x^2)^2},\\
J_3(x) &= \frac{-2}{\pi}\frac{x^3}{(1+x^2)^2}.
\end{aligned}
\tag{2.1.3}
$$

Then a straightforward calculation shows

$$\int_{-\infty}^{\infty} J_0(x)\,dx = \int_{-\infty}^{\infty} J_2(x)\,dx = 1 \tag{2.1.4}$$

and

$$\int_{-\infty}^{\infty} J_1(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} J_3(x)\,dx = 0.$$

2.2. *The Half-Space Solutions*

We next consider the upper-half-space problem:

$$
\begin{aligned}
&\Delta^2 V(x, y) = 0, \qquad y > 0,\\
&\lim_{y\to 0^+} V_{xx}(x, y) = \phi(x) \in L^q(\mathbb{R}),\\
&\lim_{y\to 0^+} V_{xy}(x, y) = \psi(x) \in L^q(\mathbb{R}).
\end{aligned}
\tag{2.2.1}
$$

This is the problem in Theorem (1.2.1) where Ω is the upper half-plane, $x_s(Q) = 1$, $y_s(Q) = 0$ and the convergence of $\mathbf{L}_{(\mathbf{n}_Q)}V$ is along lines perpendicular to the x axis.

If we compute the stress potential, we get for $X = (x, y)$ and $P = (u, 0)$ that

$$\mathscr{L}^{*}_{-\phi}\rightarrow(X) = [(J_0)_y^{*}\,\phi(x) + (J_1)_y^{*}\,\psi(x)]\,\mathbf{i} + [(J_1)_y^{*}\,\phi(x) + (J_2)_y^{*}\,\psi(x)]\,\mathbf{j}, \tag{2.2.2}$$

where $(J_k)_y(x) = (1/y)(J_k)(x/y)$, $k = 0, 1, 2$, and "$*$" denotes convolution. From (2.1.4) and the properties of approximate identities we have that for $(\phi, \psi) \in L^q(\mathbb{R}) \times L^q(\mathbb{R})$, $1 \leqslant q < \infty$,

$$\lim_{y\to 0^+} \mathscr{L}^{*}_{-\phi}\rightarrow(X) = \phi(x)\,\mathbf{i} + \psi(x)\,\mathbf{j} \qquad \text{a.e.} \tag{2.2.3}$$

and in $L^q(\mathbb{R}) \times L^q(\mathbb{R})$.

We next consider a half plane obtained from the standard one by counterclockwise rotation through an angle θ. For convenience we set $\alpha = \cos\theta$, $\beta = \sin\theta$ and consider new coordinates (x', y') where $x = \alpha x' - \beta y'$, $y = \beta x' + \alpha y'$. In the new coordinate system we consider the stress potential where the boundary $\partial\Omega = \{(x', y') \colon y' = 0\}$. Then, a straightforward computation shows that in the new coordinate system, $\mathbf{n}_Q = \beta\mathbf{i} - \alpha\mathbf{j}$ for $Q \in \partial\Omega$, so that for $X = (x', y')$, $P = (u', v')$,

$$l_{11}(P, X; \mathbf{n}_Q) = \frac{2}{\pi}\frac{\beta^2(u'-x')^2(v'-y') + 2\alpha\beta(u'-x')(v'-y')^2 + \alpha^2(v'-y')^3}{((u'-x')^2 + (v'-y')^2)^2},$$

$$l_{12}(P, X; \mathbf{n}_Q) = \frac{-2}{\pi}\frac{\begin{pmatrix}\alpha\beta(u'-x')^2(v'-y') \\ \qquad + (\alpha^2 - \beta^2)(u'-x')(v'-y')^2 - \alpha\beta(v'-y')^3\end{pmatrix}}{((u'-x')^2 + (v'-y')^2)^2},$$

$$l_{21} = l_{12},$$

$$l_{22}(P, X'; \mathbf{n}_Q) = \frac{2}{\pi}\frac{\alpha^2(u'-x')^2(v'-y') - 2\alpha\beta(u'-x')(v'-y')^2 + \beta^2(v'-y')^3}{((u'-x')^2 + (v'-y')^2)^2}. \tag{2.2.4}$$

If we assume $P \in \partial\Omega$, then $P = (u', 0)$. Writing the stress potential in (x', y') coordinates, letting $X = (x', y')$ and $\boldsymbol{\phi} = \phi(x')\,\mathbf{i} + \psi(x')\,\mathbf{j}$ we get

$$\mathscr{L}^*_{\vec{\phi}}(X) = [(K_0)_{y'}^* \phi(x) + (K_1)_{y'}^* \psi(x)]\,\mathbf{i} + [(K_1)_{y'}^* \phi(x) + (K_2)_{y'}^* \psi(x)]\,\mathbf{j}, \tag{2.2.5}$$

where

$$\begin{aligned} K_0 &= \alpha^2 J_0 + 2\alpha\beta J_1 + \beta^2 J_2, \\ K_1 &= -\alpha\beta J_0 + (\alpha^2 - \beta^2)\,J_1 + \alpha\beta J_2, \\ K_2 &= \beta^2 J_0 - 2\alpha\beta J_1 + \alpha^2 J_2. \end{aligned} \tag{2.2.6}$$

Using (2.1.4) we see that

$$\int_{-\infty}^{\infty} K_0(x)\,dx = \int_{-\infty}^{\infty} K_2(x)\,dx = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} K_1(x)\,dx = 0 \tag{2.2.7}$$

Thus we can conclude from the properties of approximate identities that for $(\phi, \psi) \in L^q(\partial\Omega) \times L^q(\partial\Omega)$,

$$\lim_{y' \to 0^+} \mathscr{L}^*_{\vec{\phi}}(x', y') = \phi(x')\,\mathbf{i} + \psi(y')\,\mathbf{j}. \tag{2.2.8}$$

It is interesting to note that the kernel $(J_3)_y(x)$ never appears. $(J_3)_y$ is the only 3rd derivative of $\tilde{F}$ which is not the dilation of an integrable function. It is somewhat analogous to the conjugate Poisson kernel.

2.3. *The Stress Kernels: Boundary Estimates*

In this section we make the basic estimates for the stress potential on and near the boundary. To facilitate the estimates it is useful to describe the component kernels in what we call geometric coordinates. For $Q \in \partial\Omega$ we consider a coordinate system $x'\mathbf{T}_Q + y'\mathbf{N}_Q$ where $\mathbf{T}_Q = x_s(Q)\,\mathbf{i} + y_s(Q)\,\mathbf{j}$ and $\mathbf{N}_Q = -y_s(Q)\,\mathbf{i} + x_s(Q)\,\mathbf{j}$. $\mathbf{T}_Q$ is the unit tangent and $\mathbf{N}_Q$ is the unit *inner* normal (N, as used here, distinguishes it from the unit outer normal denoted by n). In this coordinate system if $X = (x', y')$ and $P = (u', v')$ then $u' - x' = \langle P - X, \mathbf{T}_Q \rangle$, $v' - y' = \langle P - X, N_Q \rangle$, and $((u'-x')^2 + (v'-y')^2)^2 = |P - X|^4$. We then write the stress matrix, (the kernel of the stress potential) as

$$l^*(P, X; \mathbf{n}_Q) = \frac{2}{\pi |P-X|^4} \left\{ \begin{aligned} &\times \langle P-X, \mathbf{N}_Q \rangle \langle P-X, \mathbf{T}_Q \rangle^2 \begin{bmatrix} \beta^2 & -\alpha\beta \\ -\alpha\beta & \alpha^2 \end{bmatrix} \\ &+ \langle P-X, \mathbf{N}_Q \rangle^2 \langle P-X, \mathbf{T}_Q \rangle \begin{bmatrix} 2\alpha\beta & \beta^2-\alpha^2 \\ \beta^2-\alpha^2 & -2\alpha\beta \end{bmatrix} \\ &+ \langle P-X, \mathbf{N}_Q \rangle^3 \begin{bmatrix} \alpha^2 & \alpha\beta \\ \alpha\beta & \beta^2 \end{bmatrix}, \end{aligned} \right. \tag{2.3.1}$$

where $\alpha = x_s(Q)$ and $\beta = y_s(Q)$. To study the stress potentials we need only study the potentials

$$T_{f,i}(X) = \frac{2}{\pi} \frac{\langle P-X, \mathbf{T}_Q \rangle^{3-i} \langle P-X, \mathbf{N}_Q \rangle^i}{|P-X|^4} f(P)\, ds(P), \tag{2.3.2}$$

where $f \in L^p(\partial\Omega)$, $1 < P < \infty$, and $i = 1, 2, 3$. To study $T_{f,i}(X)$ we introduce some boundary operators.

Definition (2.3.3). Let $f \in L^p(\partial\Omega)$, $1 < p < \infty$ we define for $Q \in \partial\Omega$,

$$\begin{aligned} T_{i,\varepsilon} f(Q) &= \frac{2}{\pi} \int_{|P-Q| > \varepsilon} \frac{\langle P-Q, \mathbf{T}_Q \rangle^{3-i} \langle P-Q, \mathbf{N}_Q \rangle^i}{|P-Q|^4} f(P)\, ds(P), \\ T_i f(Q) &= \lim_{\varepsilon \to 0} T_{i,\varepsilon} f(Q), \\ MT_i f(Q) &= \sup_{\varepsilon > 0} |T_{i,\varepsilon} f(Q)|. \end{aligned} \tag{2.3.4}$$

THEOREM (2.3.5). *For $f \in L^p(\partial\Omega)$, $Q \in \partial\Omega$, $\Gamma(Q)$ the interior cone at Q, we have*

(i) *$T_i f(Q)$ exists a.e., MT_i and T_i are bounded from $L^p(\partial\Omega) \to L^p(\partial\Omega)$ and T_i is compact from $L^p(\partial\Omega) \to L^p(\partial\Omega)$.*

(ii) $$\lim_{X \to Q} T_{f,i}(X) = \begin{cases} -f(Q) + T_i f(Q), & X \in \Gamma(Q), \quad i = 1, 3 \\ T_i f(Q), & X \in \Gamma(Q), \quad i = 0. \end{cases}$$

Proof. The proof is an application of Calderón's theorem on the Cauchy integral along Lipschitz curves. To show part (ii) we make use of the following extension of Theorem 2 of Calderón's paper [2] to a wider class of operators.

THEOREM (2.3.6). *Let $F(z, t)$ be analytic in z for $|z| < R$ and bounded and measurable in t. Let ϕ be a real valued Lipschitz function on $\mathbb{R}$. For $\varepsilon > 0$ and $f \in L^p(\mathbb{R})$, $1 \leqslant p < \infty$, let*

$$L_\varepsilon f(t) = \int_{|s-t| > \varepsilon} F\left(\frac{\phi(s) - \phi(t)}{s - t}, t\right) \frac{f(s)}{s - t}\, ds. \tag{2.3.7}$$

Then there exists an absolute constant α_0 such that $\|\phi'\|_\infty \leqslant R\alpha_0(1 + \alpha_0^2)^{-1/2}$ implies that the operator $L^* f(t) = \sup_{\varepsilon > 0} |L_\varepsilon f(t)|$ is of weak type $(1, 1)$ and strong type (p, p) for $1 < p < \infty$.

Furthermore, if we set

$$G_F(\phi, f) = \left\{ t\colon \lim_{\varepsilon \to 0} \int_{|s-t| > \varepsilon} F\left(\frac{\phi(s) - \phi(t)}{s - t}, t\right) \frac{f(s)}{s - t}\, ds \text{ exists} \right\},$$

$A_R = \{F(z, t)$: F is analytic for $|z| < R$ and bounded and measurable in $t\}$ and $G(\phi, f) = \bigcap_{F \in A_R} G_F(\phi, f)$, we have that $m(\mathbb{R} \setminus G(\phi, f)) = 0$, where m is Lebesque measure on $\mathbb{R}$ and $\|\phi'\|_\infty < R\alpha_0(1 + \alpha_\alpha^2)^{-1/2}$.

Proof. Following the proof of Theorem 2 of [2],

$$L_\varepsilon f(t) = \int_{|s-t| > \varepsilon} F\left(\frac{\phi(s) - \phi(t)}{s - t}, t\right) \frac{f(s)}{s - t}\, ds$$

$$= \frac{1}{2\pi i} \int_{|z| = \rho} F(z, t)\, A_{z,\varepsilon} f(t) \frac{dz}{z}, \tag{2.3.8}$$

where ρ is sufficiently close to R and

$$A_{z,\varepsilon} f(t) = \int_{|s-t| > \varepsilon} \frac{f(s)}{(s - t) - \frac{1}{z}(\phi(s) - \phi(t))}\, ds.$$

The rest of the proof follows exactly as in Calderón's paper if we observe that $F(z, t)$ is bounded and measurable in t is all we need to complete the argument... It follows that $G(\phi, F) \supset \{t\colon \lim_{\varepsilon \to 0} A_{z,\varepsilon} f(t)$ exists$\}$

and the second part of the theorem follows because $m(\mathbb{R}\backslash\{t\colon \lim_{\varepsilon\to 0} A_{z,\varepsilon}f(t) \text{ exists}\}=0$.

Proof of Theorem (2.3.5). The proof of Theorem (2.3.5) is similar to the proof of estimates for the double layer potential in [6] and the multiple layer potential in [3]. The proof is divided into four basic steps: boundary operators, point-wise limits for stress potentials with smooth densities, maximal estimates for the components of the stress potential and extension of point-wise limits to stress potentials with L^q data.

First we obtain L^p estimates for the maximal operator $MT_i f$ by using a partition of unity, introducing local coordinates and showing that locally $T_{i,\varepsilon}f$ is bounded independent of ε by the Hardy–Littlewood maximal function of f plus a sum of Calderón operators. The compactness of the operators T_i follows from the presence of the term $P_2(\phi; x, y)=\phi(x)-\phi(y)-\phi'(y)(x-y)$ in the numerators of their kernels when the kernels are expressed in local coordinates.

Second, we obtain the almost everywhere convergence at the boundary (part (ii) of Theorem 2.3.5) in the case that f is continuously differentiable. Here the jump relations are not obtained from a relevant Green's formula but from the fact that the potentials behave, in part, like approximate identities as the point X approaches the boundary nontangentially.

Third, L^p bounds for the nontangential maximal function are obtained in terms of Calderón operators and the Hardy–Littlewood maximal function of f. Finally standard arguments (see Fabes, Jodeit, and Riviere, p. 173 [6] for details), extend the estimates for part (ii) of Theorem (2.3.5) to all $f\in L^p(\partial\Omega)$, $1<p<\infty$.

We begin by covering the set $\{X\colon \operatorname{dist}(X,\Omega)\leqslant\delta\}$ by a family of balls $\{B_j\}_{j=1}^N$ of radius r_j, centered at points $P_j\in\partial\Omega$ with the property that $B(P_j, 4r_j)\cap\Omega=B(P_j, 4r_j)\cap\{(z,w)\colon\ w>\phi_j(z)$, where $\phi_j\in C_0^1(\mathbb{R})$, $\phi_j'(0)=\phi_j(0)=0$, and $P_j=(0,0)\}$.

Let $\{n_j\}_{j=i}^N$ be a smooth partition of unity subordinate to the cover $\{B_j\}_{j=i}^N$. Let

$$K_i(P,Q)=\frac{2}{\pi}\frac{\langle P-Q, T_Q\rangle^{3-i}\langle P-Q, N_Q\rangle^i}{|P-Q|^4},\qquad i=1,2,3.$$

We can then write

$$\begin{aligned}T_{i,\varepsilon}f(Q)&=\int_{|P-Q|>\varepsilon}K_i(P,Q)\,f(P)\,ds(P)\\&=\sum_{j=1}^N\int_{|P-Q|>\varepsilon}\eta_j(P)\,K_i(P,Q)\,f(P)\,ds(P)\\&=\sum_{j=1}^N\int_{(z-W)^2+(\phi_j(z)-\phi_j(w))^2>\varepsilon^2}k_{i,j}(z,w)\,f_j(z)\,dz,\end{aligned}\qquad(2.3.10)$$

where $f_j(z)=\zeta_j(z, \phi_j(z))\, f(z, \phi_j(z))(1+\phi_j'(z)^2)^{1/2}$ and

$$k_{i,j}(z, w)=\frac{2}{\pi}\frac{1}{(1+\phi_j'(w)^2)^{3/2}}$$
$$\times\frac{((z-w)+(\phi_j(z)-\phi_j(w))\,\phi_j'(w))^{3-i}(P_2(\phi_j; z, w))^i}{((z-w)^2+(\phi_j(z)-\phi(w))^2)^2}. \qquad (2.3.11)$$

We next break up the regions of integration and get

$$\int_{(x-z)^2+(\phi_j(z)-\phi_j(w))^2>\varepsilon^2} k_{i,j}(z, w)\, f_j(z)\, dz$$
$$=\int_{|x-z|>\varepsilon} k_{i,j}(z, w)\, f_j(z)\, dz-\int_{\substack{(x-z)^2+(\phi_j(z)-\phi_j(w))^2>\varepsilon^2\\ |x-z|\leqslant\varepsilon}} k_{i,j}(z, w)\, f_j(z)\, dz$$
$$=I(\varepsilon)+II(\varepsilon). \qquad (2.3.12)$$

A standard argument shows that

$$\sup_{\varepsilon>0}|II(\varepsilon)|\leqslant cf_j^*(w), \qquad (2.3.13)$$

where

$$f_j^*(w)=\sup_{h>0}\frac{1}{2h}\int_{-h}^{h}|f_j(w+t, \phi_j(w+t))|\, dt. \qquad (2.3.14)$$

We note that $\|f_j^*\|_{L^p(\mathbb{R})}\leqslant c\|f_j\|_{L^p(\mathbb{R})}$ and that $\sum_{j=1}^N \|f_j\|_{L^p(\mathbb{R})}\approx\|f\|_{L^p(\partial\Omega)}$. Furthermore since there are only a finite number of j's we can ignore the dependence on j. We also can incorporate the term $(1+\phi'(w)^2)^{-3/2}$ into the constant which will bound the operator so we need only consider kernels

$$k_i(z, w)=\frac{2}{\pi}\frac{((z-w)+(\phi(z)-\phi(w))\,\phi'(w))^{3-i}(P_2(\phi; z, w))^i}{((z-w)^2+(\phi(z)-\phi(w))^2)^2}.$$

We will study the operators $K_{i,\varepsilon}f(w)=\int_{|z-w|>\varepsilon} k_i(z, w)\, f(z)\, dz$,

$$MK_if(w)=\sup_{\varepsilon>0}|K_{i,\varepsilon}f(w)| \qquad \text{and} \qquad K_if(w)=\lim_{\varepsilon\to 0}K_{i,\varepsilon}f(w).$$

We next introduce the Calderón operators

$$\mathscr{F}_{l,\varepsilon}f(w)=\int_{|z-w|>\varepsilon} F_l\left(\frac{\phi(z)-\phi(w)}{z-w}\right)\frac{f(z)}{z-w}\, dz,$$
$$\mathscr{M}\mathscr{F}_lf(w)=\sup_{\varepsilon>0}|\mathscr{F}_{l,\varepsilon}f(w)|, \qquad (2.3.15)$$

and

$$\mathscr{F}_l f(w) = \lim_{\varepsilon \to 0} \mathscr{F}_{l,\varepsilon} f(w),$$

where $F_l(z) = z^l/(1+z^2)^2$. Evaluating the kernels $k_i(z, w)$ we have

$$\begin{aligned}
k_1(z, w) &= \frac{2}{\pi} \frac{((z-w) + (\phi(z)-\phi(w))\, \phi'(w))^2\, P_2(\phi; z, w)}{((z-w)^2 + (\phi(z)-\phi(w))^2)^2} \\
&= \frac{2}{\pi} \frac{1}{z-w} \left\{ \phi'(w)\, F_0\left(\frac{\phi(z)-\phi(w)}{z-w}\right) \right. \\
&\quad + (1 + 2\phi'(w))^2\, F_1\left(\frac{\phi(z)-\phi(w)}{z-w}\right) \\
&\quad + (2\phi'(w) + \phi'(w)^2)\, F_2\left(\frac{\phi(z)-\phi(w)}{z-w}\right) \\
&\quad \left. + \phi'(w)^2\, F_3\left(\frac{\phi(z)-\phi(w)}{z-w}\right) \right\},
\end{aligned} \tag{2.3.16}$$

$$\begin{aligned}
k_2(z, w) &= \frac{2}{\pi} \frac{((z-w) + (\phi(z)-\phi(w))\, \phi'(w))(P_2(\phi; z, w))^2}{((z-w)^2 + (\phi(z)-\phi(w))^2)^2} \\
&= \frac{2}{\pi} \frac{1}{z-w} \left\{ \phi'(w)\, F_0\left(\frac{\phi(z)-\phi(w)}{z-w}\right) \right. \\
&\quad + (2\phi'(w) + \phi'(w)^2)\, F_1\left(\frac{\phi(z)-\phi(w)}{z-w}\right) \\
&\quad + (1 + 2\phi'(w)^2)\, F_2\left(\frac{\phi(z)-\phi(w)}{z-w}\right) \\
&\quad \left. + (\phi'(w)\, F_3\left(\frac{\phi(z)-\phi(w)}{z-w}\right) \right\},
\end{aligned} \tag{2.3.17}$$

$$\begin{aligned}
k_3(z, w) &= \frac{2}{\pi} \frac{(P_2(\phi; z, w))^3}{((z-w)^2 + (\phi(z)-\phi(w))^2)^2} \\
&= \frac{2}{\pi} \frac{1}{(z-w)} \left\{ \phi'(w)^3\, F_0\left(\frac{\phi(z)-\phi(w)}{z-w}\right) \right. \\
&\quad + 3\phi'(w)^2\, F_1\left(\frac{\phi(z)-\phi(w)}{z-w}\right) \\
&\quad + 3\phi'(w)\, F_2\left(\frac{\phi(z)-\phi(w)}{z-w}\right) \\
&\quad \left. + F_3\left(\frac{\phi(z)-\phi(w)}{z-w}\right) \right\}.
\end{aligned} \tag{2.3.18}$$

The operators $\mathscr{F}_{i,\varepsilon}$, $\mathscr{M}\mathscr{F}_i$, and $\mathscr{F}_i$ are Calderón operators and therefore bounded from $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$ for $1 < p < \infty$. Since the operators $K_{i,\varepsilon}$, MK_i and K_i are sums of Calderón operators, they also are bounded from $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$. This suffices to prove that the boundary operators $T_{i,\varepsilon}$, MT_i, and T_i are bounded from $L^p(\partial\Omega)$ to $L^p(\partial\Omega)$ for $1 < p < \infty$.

To show compactness let $\{\phi_n\}_{n=1}^{\infty}$ be a sequence of functions in $C_0^{\infty}(\mathbb{R})$ such that $\|\phi_n - \phi\|_{\infty} + \|\phi_n' - \phi'\|_{\infty} \to 0$ as $n \to \infty$. Let

$$k_{i,n}(z, w) = \frac{2}{\pi} \frac{P_2(\phi_n; z, w)\, P_2(\phi; z, w)^{i-1}((z-w) + (\phi(z) - \phi(w))\, \phi'(w))^{3-i}}{((z-w)^2 + (\phi(z) - \phi(w))^2)^2}.$$

Let $K_{i,n}f(w) = \int_{\mathbb{R}} k_{i,n}(z, w)\, f(z)\, dz$. Since $\phi_n \in C_0^{\infty}$, $k_{i,n}(z, w)$ is bounded so that $K_{i,n}f(w)$ exists and is compact from $L^p(\mathbb{R}) \to L^p(\mathbb{R})$. Next note

$$\begin{aligned}
&(K_{i,n} - K_i)\, f(w) \\
&= \text{p.v.} \int_{\mathbb{R}} (k_{i,n}(z, w) - k_i(z, w))\, f(z)\, dz \\
&= \frac{2}{\pi}\, \text{p.v.} \int \frac{P_2(\phi_n - \phi; z, w)\, P_2(\phi; z, w)^{i-1}(z - w + (\phi(z) - \phi(w))\, \phi'(w))^{3-i}\, dz}{((z-w)^2 + (\phi(z) - \phi(w))^2)^2}.
\end{aligned} \tag{2.3.19}$$

By Calderón's Theorem K_i is then the norm limit of the compact $K_{i,n}$'s and so is itself compact.

We now assume that f is continuously differentiable on $\partial\Omega$. For $X \in \Gamma(Q)$, the interior cone with vertex at Q, we consider the potential

$$T_{f,i}(X) = \int_{\partial\Omega} k_i(P, X, Q)\, f(P)\, ds(P), \tag{2.3.20}$$

where $k_i(P, X, Q) = (2/\pi)(\langle P - X, \mathrm{T}_Q\rangle^{3-i} \langle P - X, N_Q\rangle^i / |P - X|^4)$, $i = 1, 2, 3$. (Note that the kernel $k_i(P, Q)$ just considered is the same as $k_i(P, Q, Q)$.)

Given a $\delta > 0$, we have

$$\begin{aligned}
T_{f,i}(X) &= \int_{|P-Q|>\delta} k_i(P, X, Q)\, f(P)\, ds(P) + \int_{|P-Q|\leqslant\delta} k_i(P, X, Q)\, f(P)\, ds(P) \\
&= \int_{|P-Q|>\delta} [k_i(P, X, Q) - k_i(P, Q)]\, f(P)\, ds(P) \\
&\quad + \int_{|P-u|>\delta} k_i(P, Q)\, f(P)\, ds(P) - \text{p.v.} \int k_i(P, Q)\, f(P)\, ds(P) \\
&\quad + \int_{|P-Q|\leqslant\delta} k_i(P, X, Q)(f(P) - f(Q))\, ds(P) \\
&\quad + f(Q) \int_{|P-Q|\leqslant\delta} k_i(P, X, Q)\, ds(P) + \text{p.v.} \int k_i(P, Q)\, f(P)\, ds(P).
\end{aligned}$$

⑤

Since $\lim_{\delta\to 0} T_{i,\delta} f(Q)$ exists a.e., we know that for $\varepsilon > 0$ and almost every $Q \in \partial\Omega$ we can find a $\delta_1 > 0$ so that

$$\left| \int_{|P-Q|>\delta_1} k_i(P,Q)\, f(P)\, ds(P) - \text{p.v.} \int_{\partial\Omega} k_i(P,Q)\, f(P)\, ds(P) \right| < \varepsilon. \tag{2.3.22}$$

Since f is continuously diffentiable, $k_i(P,X,Q)(f(Q))$ is bounded. Thus we can choose $\delta_2 > 0$ so that

$$\left| \int_{|P-Q|\leqslant\delta} k_i(P,X,Q)(f(P)-f(Q))\, ds(P) \right| < \varepsilon \tag{2.3.23}$$

whenever $\delta \leqslant \delta_2$.

Given a $\delta > 0$, fixed, we can find $\delta_3 > 0$ so that

$$\left| \int_{|P-Q|>\delta} (k_i(P,X,Q) - k_i(P,Q))\, f(P)\, ds(P) \right| < \varepsilon \tag{2.3.24}$$

whenever $|X-Q| < \delta_3$.

If we set $\delta = \min\{\delta_1, \delta_2\}$ we have shown that for $\varepsilon > 0$, we can find a $\delta_3 > 0$ such that

$$\left| T_{f,i}(X) - f(Q) \int_{|P-Q|\leqslant\delta} k_i(P,X,Q)\, ds(P) - \text{p.v.} \int k_i(P,Q)\, f(P)\, ds(P) \right| < 3\varepsilon, \tag{2.3.25}$$

whenever $|X-Q| < \delta_3$.

The problem now is to evaluate $\int_{|P-Q|\leqslant\delta} k_i(P,X,Q)\ ds(P)$. This problem is avoided in the case of the double layer potential by a Green's formula argument. However, for this potential we have the following substitute:

Lemma (2.3.26).

$$\lim_{\delta\to 0} \lim_{\substack{X\to Q\\ X\in\Gamma(Q)}} \int_{|P-Q|\leqslant\delta} k_i(P,X,Q)\, ds(P) = \begin{cases} -1, & i=1,3 \\ 0, & i=2 \end{cases}$$

for almost every $Q \in \partial\Omega$.

Proof. We begin by choosing $\delta > 0$ and a ball $B = B(Q,\delta)$ so that $B \cap \Omega = B \cap \{(z,w)\colon w > \phi(z),\ \phi \in C_0^1(\mathbb{R}),\ \phi(0) = \phi'(0) = 0$ and $\|\phi'\|_\infty$ small enough to permit the use of Calderón's theorem$\}$. Then consider $(z,w)_\circ$ coordinates where the z axis is the line tangent to $\partial\Omega$ at the point Q.

For $P \in \partial\Omega \cap B$ and $X = Q + w\mathbf{N}_Q$, ($\mathbf{N}_Q$ is the unit inner normal at Q), we have $P=(z,\phi(z))$, $X=(0,w)$, $\mathbf{T}_Q=(1,0)$ and $\mathbf{N}_Q=(0,1)$. For these points,

$$k_i(P,X,Q)=\frac{2}{\pi}\frac{z^{3-i}(\phi(z)-w)^i}{(z^2+(\phi(z)-w)^2)^2}. \tag{2.3.27}$$

We begin by noting that

$$\lim_{w\to 0}\frac{2}{\pi}\int_{|z|\leqslant\delta}\frac{z^{3-i}w^i}{(z^2+w^2)^2}\,dz=\begin{cases}1, & i=0,2\\ 0, & i=1.\end{cases} \tag{2.3.28}$$

This together with (2.3.23) tells us that if $\partial\Omega$ coincides with its tangent line for $|P-Q|\leqslant\delta$ then,

$$\lim_{w\to 0}\int_{|P-Q|\leqslant\delta}k_i(P,Q+w\mathbf{N}_Q,Q)\,ds(P)=\begin{cases}-1, & i=1,3\\ \ \ 0, & i=2\end{cases} \tag{2.3.29}$$

since $\phi(z)=0$ for small values of z.

We next set $X=Q+w\mathbf{N}_Q$ and assume $w<\delta$.

$$\begin{aligned}\int_{|P-Q|\leqslant\delta}&k_i(P,X,Q)\,ds(P)\\ &=\frac{2}{\pi}\int_{|z|\leqslant\delta}\frac{z^{3-i}(\phi(z)-w)^i}{(z^2+\phi(z)-w)^2)^2}\,dz-\frac{2}{\pi}\int_{\substack{|z|\leqslant\delta\\ z^2+\phi(z)^2>\delta^2}}\frac{z^{3-i}(\phi(z)-w)^i}{(z^2+(\phi(z)-w)^2)^2}\,dz\\ &=I(\delta)+II(\delta).\end{aligned} \tag{2.3.30}$$

To estimate these integrals we introduce $\eta(\delta)=\sup_{|z|\leqslant\delta}|\phi'(z)|$. Since $\phi'(z)\to 0$ as $z\to 0$ we have $\eta(\delta)\to 0$ as $\delta\to 0$. Then if $|z|\leqslant\delta$ and $z^2+\phi(z)^2>\delta^2$ we have $z^2(1+\eta(\delta)^2)>\delta^2$ which implies that $\delta/(1+\eta(\delta)^2)^{1/2}<|z|<\delta$. This means that the measure of the region of integration in $II(\delta)$ is less than $\delta(1-(1/\sqrt{1+\eta(\delta)^2})=o(\delta)$ as $\delta\to 0$. In this region

$$\begin{aligned}\left|\frac{z^{3-i}(\phi(z)-w)}{(z^2+(\phi(z)-w)^2)^2}\right|&\leqslant c\,\frac{\delta^{3-i}(\eta(\delta)\,\delta+w)^i}{\delta^4}\\ &\leqslant\frac{c}{\delta}.\end{aligned} \tag{2.3.31}$$

Hence,

$$|II(\delta)|\leqslant c\delta(1-(1/\sqrt{1+\eta(\delta)^2}))\frac{1}{\delta}\to 0\qquad\text{as}\quad\delta\to 0. \tag{2.3.32}$$

We now must estimate $I(\delta)$. We consider

$$
\begin{aligned}
I(\delta) - \frac{2}{\pi}\int_{|z|\leqslant\delta} \frac{z^{3-i}(-w)^i}{(z^2+(\phi(z)-w)^2)^2}\,dz
&= \sum_{j=0}^{i-1} (-1)^j \frac{2}{\pi}\int_{|z|\leqslant\delta} \frac{z^{3-i}\binom{i}{j}\,\phi(z)^{i-j}\,w^j}{(z^2+(\phi(z)-w)^2)^2}\,dz \\
&= \sum_{j=0}^{i-1} (-1)^j \frac{2}{\pi}\binom{i}{j}\left\{\int_{|z|<w/2} \frac{z^{3-i}\phi(z)^{i-j}\,w^j}{(z^2+(\phi(z)-w)^2)^2}\,dz \right. \\
&\quad \left. + \int_{w/2<|z|\leqslant\delta} \frac{z^{3-i}\phi(z)^{i-j}w^j}{(z^2+(\phi(z)-w)^2)^2}\,dz\right\}. \qquad (2.3.33)
\end{aligned}
$$

Recall we are assuming $w<\delta$. Then,

$$
\begin{aligned}
\left|\frac{z^{3-i}\phi(z)^{i-j}w^j}{(z^2+(\phi(z)-w)^2)^2}\right| &< \frac{w^{3-i}(\eta(\delta)\,w)^{i-j}w^j}{(w/2)^4} \\
&= \frac{16\eta(\delta)^{i-j}}{w}. \qquad (2.3.34)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\left|\int_{|z|<w/2} \frac{z^{3-i}\phi(z)^{i-j}w^j}{(z^2+(\phi(z)-w)^2)^2}\,dz\right| \\
&\quad \leqslant \frac{16\eta(\delta)^{i-j}}{w}\cdot w \\
&\quad = 16\eta(\delta)^{i-j} \to 0 \qquad \text{as} \quad \delta\to 0. \qquad (2.3.35)
\end{aligned}
$$

We next consider the region $w/2<|z|\leqslant\delta$. Since $i=1,2,3$ and $0\leqslant j\leqslant i-1$, we have six cases for the numerator. We first examine the case $j>0$. If $j=2$, $i=3$ and if $j=1$, $i=2$, or $i=3$. Since $i>j$ we have

$$
\left|\frac{z^{3-i}\phi(z)^{i-j}\,w^j}{(z^2+(\phi(z)-w)^2)^2}\right| \leqslant \frac{c|z|^{3-j}\,\eta(\delta)^{i-j}\,w^j}{(z^2+w^2)^2}. \qquad (2.3.36)
$$

Hence,

$$
\begin{aligned}
&\left|\int_{w/2<|z|\leqslant\delta} \frac{z^{3-i}\phi(z)^{i-j}\,w^j}{(z^2+(\phi(z)-w)^2)^2}\,dz\right| \\
&\quad \leqslant c\left|\int_{-\infty}^{\infty} \frac{|z|^{3-j}\,w^j}{(z^2+w^2)^2}\,dz\right|\,\eta(\delta)^{i-j} \\
&\quad \to 0 \qquad \text{as} \quad \delta\to 0 \text{ since } i-j>0 \text{ and } j>0. \qquad (2.3.37)
\end{aligned}
$$

For the case $j=0$ we need Calderón's theorem to get the estimates. We begin with

$$\begin{aligned}
&\int_{w/2<|z|<\delta} \frac{z^{3-i}\phi(z)^i}{(z^2+(\phi(z)-w)^2)^2}\,dz \\
&\quad = \int_{w/2<|z|\leqslant\delta} z^{3-i}\phi(z)^i \left\{\frac{1}{(z^2+(\phi(z)-w)^2)^2} - \frac{1}{(z^2+\phi(z)^2)^2}\right\} dz \\
&\qquad + \int_{w/2<|z|\leqslant\delta} \frac{(z-0)^{3-i}(\phi(z)-\phi(0))^i}{((z-0)^2+(\phi(z)-\phi(0))^2)^2}\,dz \\
&\quad = A(\delta)+B(\delta). \qquad (2.3.38)
\end{aligned}$$

Since

$$\left|\frac{1}{(z^2+(\phi(z)-w)^2)^2} - \frac{1}{(z^2+\phi(z)^2)^2}\right| \leqslant c\,\frac{|z|^3\,w}{(z^2+w^2)^4} \qquad \text{when} \quad |z|>\frac{w}{2}$$

we have

$$\begin{aligned}
|A(\delta)| &\leqslant c\eta(\delta)^i \int_{w/2<|z|\leqslant\delta} \frac{|z|^6\,w}{(z^2+w^2)^4}\,dz \\
&< c\eta(\delta)^i \int_{-\infty}^{\infty} \frac{x^6}{(1+x^2)^4}\,dx. \qquad (2.3.39)
\end{aligned}$$

Clearly $A(\delta)\to 0$ as $\delta\to 0$.

To estimate $B(\delta)$ we use Theorem (2.3.6). For $Q'\in\partial\Omega$, let $(s, \phi_{Q'}(s))$ be the parametrization for the boundary in local coordinates with origin at Q' and with the s-axis along the tangent line to $\partial\Omega$ a Q'. Then at almost every point $Q'\in\partial\Omega$ we need to show

$$\lim_{a,b\to 0} \int_{a<|s|<b} F_i\left(\frac{\phi_{Q'}(s)}{s}\right)\frac{ds}{s} = 0, \qquad (2.3.40)$$

where F_i defined in (2.3.15). If we write (2.3.40) in terms of coordinates based at Q, then for $Q'=(x, \phi_Q(x))$ in the Q based coordinates,

$$\int_{a<|s|<b} F_i\left(\frac{\phi_{Q'}(s)}{s}\right)\frac{ds}{s} = \int_{\alpha<|x-y|<\beta} F_i\circ\psi_x\left(\frac{\phi_Q(x)-\phi_Q(y)}{x-y}\right)\frac{dy}{x-y}, \qquad (2.3.41)$$

where $\psi_x(z)=(z-\phi'(x))/(1-z\phi'(x))$ and α, $\beta\to 0$ as a, $b\to 0$.

Clearly the function $F_i\circ\psi_x(z)$ is analytic in z and bounded and measurable in x for small $\|\phi'\|_\infty$. Hence we can apply Theorem (2.3.6) to obtain (2.3.40) for almost every $Q\in\partial\Omega$. The term $B(\delta)$ arises at each Q. By what we have shown $B(\delta)\to 0$ as $\delta\to 0$ for almost every $Q\in\partial\Omega$.

We next have the estimate,

$$\begin{aligned}
&\left|\int_{|z|\leqslant\delta}\frac{z^{3-i}w^{i}}{(z^2+(\phi(z)-w)^2)^2}-\int_{|z|\leqslant\delta}\frac{z^{3-i}w^{i}}{(z^2+w^2)^2}\,dz\right|\\
&\qquad\leqslant\left|\int_{|z|<w/2}(\cdots)\,dz\right|+\left|\int_{w/2\leqslant|z|\leqslant\delta}(\cdots)\,dz\right|\\
&\qquad\leqslant c\eta(\delta)\left\{\int_{|z|<w/2}\frac{|z|^{4-i}\,w^{3+i}}{w^8}\,dz+\int_{-\infty}^{\infty}\frac{|z|^6\,w}{(z^2+w^2)^4}\,dz\right\}\\
&\qquad\to 0 \qquad \text{as}\quad \delta\to 0.
\end{aligned}\tag{2.3.42}$$

Finally we note that for $X\in\Gamma(Q)$ and $\tilde{X}=Q+\langle X-Q,\mathbf{N}_Q\rangle\,\mathbf{N}_Q$, a straightforward computation shows that

$$\lim_{\delta\to 0}\left\{\lim_{X\to Q}\int_{|P-Q|\leqslant\delta}k_i(P,X,Q)\,ds(Q)-\int_{|P-Q|\leqslant\delta}k_i(P,\tilde{X},Q)\,ds(Q)\right\}=0. \tag{2.3.43}$$

Since

$$\lim_{\delta\to 0}\frac{2}{\pi}\int_{|z|\leqslant\delta}\frac{z^{3-i}(-w)^{i}}{(z^2+w^2)^2}\,dz=\begin{cases}-1, & i=1,3\\ \ \ 0, & i=2\end{cases}$$

we can combine estimates (2.3.32), (2.3.35), (2.3.37), (2.3.39), (2.3.40), and (2.3.42) to conclude the proof of the lemma.

From (2.3.25) and Lemma (2.3.26) we conclude that for f continuously differentiable

$$\lim_{\substack{X\to Q\\ X\in\Gamma(Q)}}T_{f,i}(X)=\begin{cases}(-I+T_i)\,f(Q), & i=1,3\\ T_i f(Q), & i=2.\end{cases} \tag{2.3.44}$$

To generalize (2.3.44) to more general densities we need

LEMMA (2.3.45). *Let* $f\in L^q(\partial\Omega)$. *Let* $T^*_{f,i}(Q)=\mathrm{Sup}\{|T_{f,i}(X)|:\ X\in\Gamma^\delta_\alpha(Q)\}$. *Then* $T^*_{f,i}$ *is bounded from* $L^q(\partial\Omega)$ *into itself for* $1<q<\infty$.

Proof. The proof of this lemma is contained in the proof of Theorem (4.1.1) of Cohen and Gosselin [3]. More precisely, the operator u^*_1 defined by (4.1.7) of [3] is estimated by estimating the kernel, $k(x,t,z)=(x-z)^\alpha(t-\phi(z))^\beta/(((x-z)^2+(t-\phi(z))^2)^\gamma)$ (where $2\gamma=\alpha+\beta+1$), given in (4.1.10). The introduction of local coordinates here shows that we need to estimate the kernel $(z-u)^{3-i}(\phi(z)-u)^i/((z-u)^2+(\phi(z)-u)^2)^2)$ which is obviously a special case of $k(x,t,z)$ which appears above.

Finally, preceding as in Theorem 1.3 of [6], one uses the maximal estimates of Lemma (2.3.45) and the fact that (ii) of Theorem (2.3.5) holds for f continuously differentiable to obtain part (ii) for general $f \in L^q(\partial\Omega)$.

3. INVERTIBILITY

In this section we show that $-I+\mathscr{L}^*$ is invertible on a subset of $L^q(\partial\Omega) \times L^q(\partial\Omega)$ satisfying appropriate moment conditions. This will tell us that the boundary value problem, $\Delta^2 v=0$ in Ω, $\mathbf{L}v=\boldsymbol{\phi}$ on $\partial\Omega$, is solvable as a modified lower order potential.

An important part of the demonstration of the invertibility of $-I+\mathscr{L}^*$ is done in [4]. There it is shown that the adjoint operator $(-\bar{I}+\bar{\mathscr{L}})^*$ is invertible on the coset space $\mathscr{C}_p$. In this paper we show that the invertibility extends to the appropriate space of pairs of $L^q(\partial\Omega)$ functions.

3.1. *Previous Results*

We begin by explaining the results of [4], starting with some definitions.

DEFINITION (3.1.1). $\mathscr{C}_p^{\perp} = \{\boldsymbol{\phi} = (\phi, \psi) \in L^q(\partial\Omega) \times L^q(\partial\Omega): \langle \bar{g}, \boldsymbol{\phi} \rangle = \int_{\partial\Omega} g(Q)\, \phi(Q) + h(Q)\, \psi(Q)\, ds(Q)=0$ for all $\bar{g} \in \mathscr{C}_p\}$

Remark (3.1.2). $\mathscr{C}_p^* = L^q(\partial\Omega) \times L^q(\partial\Omega)/\mathscr{C}_p^{\perp}$

Remark (3.1.3). Since $\mathscr{C}_p$ consists of pairs of $L^p(\partial\Omega)$ functions satisfying $\int_{\partial\Omega} gx_s + hy_s\, ds=0$, it follows that $\mathscr{C}_p^{\perp} = \{\lambda(x_s\mathbf{i}+y_s\mathbf{j}): \lambda \in \mathbb{R}\}$. The elements of $\mathscr{C}_p^*$ are cosets of the form $\bar{\phi}=\boldsymbol{\phi}+\lambda\mathrm{T}_Q$ where $\mathrm{T}_Q=x_s(Q)\,\mathbf{i}+y_s(Q)\,\mathbf{j}$ and $\lambda \in \mathbb{R}$.

For $\bar{g} \in \mathscr{C}_p$, let $u_m(\bar{g}; X)$ be the modified multiple layer potential with density $\bar{g}$ (see (1.1.2)). Define $A(P)=\int_{P_0}^{P} g(Q)\, x_s(Q)+h(Q)\; y_s(Q)\, ds(Q)$, where P, $P_0 \in \partial\Omega$, P_0 is fixed and the integration proceeds counterclockwise around $\partial\Omega$. Let $\dot{A}=(A, g, h)$ and let $u(\dot{A}; X)$ be the multiple layer potential with density $\dot{A}$ (see Sect. 2.5 of [4] for further details about the multiple layer potential). Then for $X \in \Gamma(Q)$,

$$\lim_{X \to P} \nabla u_m(\bar{g}; X) = \lim_{X \to P} \nabla u(\dot{A}, X) = \pi(-I+\mathscr{K})\, \dot{A}(P), \qquad (3.1.4)$$

where π is defined by $\pi\dot{A}=\pi(A, g, h)=(g, h)$. The operator $\mathscr{K}$ is defined in (1.2.3) of [4]. We then define $\bar{I}$ and $\bar{\mathscr{L}}$ by $\bar{I}\bar{g}=\pi\dot{A}$ and $\bar{\mathscr{L}}\bar{g}=\pi\mathscr{K}\dot{A}$. Then (1.2.4) of [4] implies that $\nabla u_m(X) \to (-\bar{I}+\mathscr{L})\, \bar{g}$ as $X \to P$ nontangentially from the interior. The operator $(-\bar{I}+\bar{\mathscr{L}})^*$ is the adjoint of $(-\bar{I}+\bar{\mathscr{L}})$ and it acts on the coset space $\mathscr{C}_p^*$.

We now introduce some more spaces.

DEFINITION (3.1.5). Let $\bar{W}_+ = \mathrm{Ker}(-\bar{I}+\bar{\mathscr{L}})^*$ and $\bar{V}_+ = \mathrm{Ker}(-\bar{I}+\bar{\mathscr{L}})$.

DEFINITION (3.1.6). Let $\nabla\mathscr{I} = \{\alpha(x, y) + \beta(1, 0) + \gamma(0, 1)\colon \alpha, \beta, \gamma$, are real$\}$. The annihilator of $\nabla\mathscr{I}$ is then $\nabla\mathscr{I}^{\perp} = \{\bar{\phi} \in \mathscr{C}_p^*\colon \langle \bar{g}, \bar{\phi} \rangle = 0$ for all $\bar{g} \in \nabla\mathscr{I}\}$.

Remark (3.1.7). Recalling the space $\mathscr{I}(\Omega) = \{\alpha(x^2 + y^2) + \beta x + \gamma y + \delta\}$ defined in Section 1.1 of [4] we see that $\nabla\mathscr{I}$ consists of the gradients of polynomials in $\mathscr{I}$. The dual pairing $\langle \bar{g}, \bar{\phi} \rangle$ is defined by $\langle \bar{g}, \bar{\phi} \rangle = \int_{\partial\Omega} g(Q)\, \phi(Q) + h(Q)\, \psi(Q)\, ds(Q)$ where $\boldsymbol{\phi} = (\phi, \psi) \in \bar{\phi}$. This is well defined since if $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ are representatives of the same coset then $\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2 \in \mathscr{C}_p^{\perp}$.

We next want to obtain the adjoint operator $(-\bar{I} + \bar{\mathscr{L}})^*$ as the boundary value of $\mathbf{L}_{(\mathbf{n}_Q)} v_m(X)$. This is done in [4] in the following manner.

Let $\{B_j\}_{j=1}^{n}$, γ_j, P_j, ϕ_j and η_j be as in the proof of Theorem (2.3.5).

Let $\delta_0 = \frac{1}{8} \operatorname{dist}((\bigcup_{j=1}^{N} B_j)^c, \partial\Omega)$. For a pair of functions $\boldsymbol{\phi} = (\phi, \psi)$ defined in Ω let

$$(\boldsymbol{\phi})_t^-(P) = \sum_{j=1}^{N} \eta_j(P)\, \boldsymbol{\phi}(P + t\mathbf{N}_j), \tag{3.1.8}$$

where $0 < t < \delta_0$ and $\mathbf{N}_j$ is the unit inner normal at P_j. For $\bar{a} \in \mathscr{C}_p^*$ we say $\lim_{t\to 0} (\boldsymbol{\phi})_t^- = \bar{\alpha}$ if $\lim_{t\to 0} \langle \bar{g}, (\boldsymbol{\phi})_t^- \rangle = \langle \bar{g}, \bar{\alpha} \rangle$ for all $\bar{g} \in \mathscr{C}_p$. In what follows we let $\mathbf{L}v$ denote $\mathbf{L}_{(\mathbf{n}_Q)} v$.

In [4] we show that if $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ are two distinct representatives of the same coset and $v_1 = v_m(\boldsymbol{\phi}_1; X)$ and $v_2 = v_m(\boldsymbol{\phi}_2; X)$ are the corresponding modified lower order potentials, then $\langle \bar{g}, (\mathbf{L}v_1 - \mathbf{L}v_2)_t^- \rangle = 0$ for all $t > 0$ and all $\bar{g} \in \mathscr{C}_p$. We then have the following theorems:

THEOREM (3.1.9). *For $\bar{\phi} \in \mathscr{C}_p^*$, $\boldsymbol{\phi}$ a representative of $\bar{\phi}$ and $v_m(\boldsymbol{\phi}; X)$ the modified lower order potential with density $\boldsymbol{\phi}$ then* $\lim_{t\to 0} (\mathbf{L}v_m)_t^- = (-\bar{I} + \bar{\mathscr{L}})^* \bar{\phi}$ *in the sense that,*

$$\lim_{t\to 0} \langle \bar{g}, (\mathbf{L}v_m)_t^- \rangle = \langle (-\bar{I} + \bar{\mathscr{L}})\, \bar{g}, \bar{\phi} \rangle \tag{3.1.10}$$

Proof. This is theorem (3.4.2) of [4].

THEOREM (3.1.11). $\nabla\mathscr{I} = \nabla_+$ *and* $\mathscr{C}_p^* = \nabla\mathscr{I}^{\perp} \oplus \bar{W}_+$.

Proof. The proof is in Theorem (2.2.2) and (3.4.1) of [4].

COROLLARY (3.1.12). *If $\bar{\phi}_0 \in \nabla\mathscr{I}^{\perp}$ then there is a unique $\bar{\phi}_1 \in \nabla\mathscr{I}^{\perp}$ such that if $\boldsymbol{\phi}_1 \in \bar{\phi}_1$ and $v_m(\boldsymbol{\phi}_1; X)$ is the modified lower order potential with density $\boldsymbol{\phi}_1$, then*

$$\begin{aligned} &\Delta^2 v_m(\boldsymbol{\phi}_1; X) = 0, \qquad X \in \Omega, \\ &\lim_{t\to 0} (\mathbf{L}v_m)_t^- = \bar{\phi}_0. \end{aligned} \tag{3.1.13}$$

3.2. *Invertibility of* $(-I+\mathscr{L}^*)$

⑦

To conclude this paper we need to relate the "abstract" adjoint operator $(-\bar{I}+\bar{\mathscr{L}})^*$ to the pointwise almost everywhere defined integral operator $-I+\mathscr{L}^*$ obtained in (2.3.5). We use the invertibility of $(-\bar{I}+\bar{\mathscr{L}})^*$ to obtain the invertibility of $(-I+\mathscr{L}^*)$.

We begin by letting α: $L^q(\partial\Omega)\times L^q(\partial\Omega)\rightarrow \mathscr{C}_p^*$ be given by $\alpha(\boldsymbol{\phi})=\bar{\phi}=\boldsymbol{\phi}+\mathscr{C}_p^{\perp}$. Next let $T=-I+\mathscr{L}^*$ and $\bar{T}=(-\bar{I}+\bar{\mathscr{L}})^*$. Finally let $(L^q\times L^q)_0(\partial\Omega)=\{\boldsymbol{\phi}\in L^q(\partial\Omega)\times L^q(\partial\Omega)$: $\int_{\partial\Omega} g(Q) \quad \phi(Q)+h(Q) \quad \psi(Q)$ $ds(Q)=0$ for all $(g, h)\in\nabla\mathscr{I}\}$. We then have the following important result.

Lemma (3.2.1). $\alpha T\boldsymbol{\phi}=\bar{T}\alpha\boldsymbol{\phi}$ *for all* $\boldsymbol{\phi}\in(L^q\times L^q)_0(\partial\Omega)$.

Proof. Let $\boldsymbol{\phi}\in(L^q\times L^q)_0$ and let $v_m(\boldsymbol{\phi}; X)$ be the modified lower order potential with density $\boldsymbol{\phi}$. Then for $\bar{g}\in\mathscr{C}_p$,

$$\langle \bar{g}, \bar{T}\alpha\boldsymbol{\phi}\rangle=\langle \bar{g}, \bar{T}\bar{\phi}\rangle=\lim_{t\rightarrow 0}\langle \bar{g}, (\mathbf{L}v_m)_t^-\rangle. \tag{3.2.2}$$

On the other hand,

$$\begin{aligned}\langle \bar{g}, \alpha T\boldsymbol{\phi}\rangle&=\int \bar{g}(Q)\,\alpha\Big(\lim_{\substack{X\rightarrow Q\\ X\in\Gamma(Q)}}\mathbf{L}v_m(X)^{\mathrm{T}}\Big)\,ds(Q)\\ &=\sum_{j=1}^{N}\int_{\partial\Omega}\eta_j(Q)\,\bar{g}(Q)\Big(\lim_{t\rightarrow 0}\mathbf{L}v_m(\phi; Q+t\mathbf{N}_j)^{\mathrm{T}}\Big)\,ds(Q)\\ &=\lim_{t\rightarrow 0}\sum_{j=1}^{N}\int_{\partial\Omega}\eta_j(Q)\,\bar{g}(Q)\,\mathbf{L}v_m(\phi; Q+t\mathbf{N}_j)^{\mathrm{T}}\,ds(Q)\\ &=\lim_{t\rightarrow 0}\langle \bar{g}, (\mathbf{L}v_m)_t^-\rangle. \end{aligned}\tag{3.2.3}$$

By (3.2.2) and (3.2.3) $\bar{T}\alpha\boldsymbol{\phi}=\alpha T\boldsymbol{\phi}$ as elements in $\mathscr{C}_p^*$.
We next consider the modified lower order potential with density in $\mathscr{C}_p^{\perp}$.

Lemma (3.2.4). *If* $\boldsymbol{\phi}\in\mathscr{C}_p^{\perp}$ *and* $v=v_m(\boldsymbol{\phi}; X)$ *then* $\lim_{X\rightarrow Q\in\partial\Omega, X\in\Gamma(Q)}$ $\mathbf{L}v_m(X)=-2\boldsymbol{\phi}(Q)$.

Proof. $\mathscr{C}_p^{\perp}=\{\lambda\mathbf{T}_Q$: $\lambda\in\mathbb{R}$, $\mathbf{T}_Q=x_s(Q)\,\mathbf{i}+y_s(Q)\,\mathbf{j}\}$. Thus we can explicitly compute

$$\begin{aligned}v_m(\lambda\mathbf{T}_Q; X)&=\int_{\alpha\Omega_0}2\tilde{F}_x(P-X)\,\lambda x_s(P)+2\tilde{F}_y\lambda y_s(P)\,ds(P)\\ &=\lambda 2\tilde{F}(P-X)\,\Big]_{P_0^+}^{P_0^-}\\ &=-\lambda|P_0-X|^2.\end{aligned}\tag{3.2.5}$$

Employing the definition of $\mathbf{L}=\mathbf{L}_{(\mathbf{n}_Q)}$ we get

$$\mathbf{L}v_m(\lambda T_Q; X) = -2\lambda \mathbf{T}_Q. \tag{3.2.6}$$

In other words, for $\boldsymbol{\phi} \in \mathscr{C}_p^{\perp}$, $(-I+\mathscr{L}^*)\,\boldsymbol{\phi} = -2\boldsymbol{\phi}$. We now have our main theorem.

THEOREM (3.2.7). *The operator* $(-I+\mathscr{L}^*)^{-1}$ *exists for* $\boldsymbol{\phi} \in (L^q \times L^q)_0(\partial\Omega)$.

Proof. Let $T = -I+\mathscr{L}^*$ and $\bar{T} = (-\bar{I}+\bar{\mathscr{L}})^*$ as before. Assume $\boldsymbol{\phi} \in (L^q \times L^q)_0(\Omega)$ and $T\boldsymbol{\phi} = \mathbf{0}$. Le $\alpha(\boldsymbol{\phi}) = \bar{\boldsymbol{\phi}} = \boldsymbol{\phi} + \mathscr{C}_p^{\perp}$ as in Lemma (3.2.1). Then $\alpha(T(\boldsymbol{\phi})) = \alpha(\mathbf{0}) = \bar{0}$. Then by Lemma (3.2.1) $\bar{T}\alpha(\boldsymbol{\phi}) = \alpha T(\boldsymbol{\phi}) = \bar{0}$. That is, $\bar{T}\bar{\boldsymbol{\phi}} = \bar{0}$. But $\bar{T}$ is 1 to 1 so that $\bar{\boldsymbol{\phi}} = \bar{0}$. That is $\bar{\boldsymbol{\phi}} = \mathscr{C}_p^{\perp}$ which implies that $\boldsymbol{\phi} \in \mathscr{C}_p^{\perp}$. By Lemma (3.2.1) we have $\boldsymbol{\phi} = -\frac{1}{2}T\boldsymbol{\phi}$ since $\boldsymbol{\phi} \in \mathscr{C}_p^{\perp}$. But by our original assumption that $T\boldsymbol{\phi} = \mathbf{0}$, we conclude $\boldsymbol{\phi} = 0$. Thus T is one to one on $(L^q \times L^q)_0(\partial\Omega)$.

We next show that T maps $(L^q \times L^q)_0(\partial\Omega)$ to itself. Let $\boldsymbol{\phi} \in (L^q \times L^q)_0(\partial\Omega)$. Then since $\bar{T}$ is 1 to 1 from $\nabla\mathscr{I}^{\perp}$ to itself, if $\bar{g} \in \nabla\mathscr{I}$ we have

$$\begin{aligned} \langle \bar{g}, T\boldsymbol{\phi} \rangle &= \langle \bar{g}, (\mathbf{L}v_m)^- \rangle \\ &= \langle \bar{g}, \bar{T}\bar{\boldsymbol{\phi}} \rangle \\ &= 0 \end{aligned} \tag{3.2.8}$$

provided $\bar{\boldsymbol{\phi}} \in \nabla\mathscr{I}^{\perp}$. But if $\boldsymbol{\phi} \in (L^q \times L^q)_0(\partial\Omega)$ then clearly $\bar{\boldsymbol{\phi}} \in \nabla\mathscr{I}^{\perp}$ since for any other representative $\boldsymbol{\phi}_1 = (\phi_1, \psi_1) \in \bar{\boldsymbol{\phi}}$ we have $\boldsymbol{\phi}_1 = \boldsymbol{\phi} + \lambda_1 \mathbf{T}_Q$ and,

$$\begin{aligned} \int_{\partial\Omega} g\phi_1 + h\psi_1 &= \int_{\partial\Omega} g(\phi + \lambda_1 x_s) + h(\psi + \lambda_1 y_s) \\ &= \int_{\partial\Omega} g\phi + h\psi \, ds + \lambda_1 \int_{\partial\Omega} g x_s + h y_s \, ds \\ &= 0 \end{aligned} \tag{3.2.9}$$

for all $\bar{g} \in \nabla\mathscr{I}$.

Finally by Theorem (2.3.5) the operator $\mathscr{L}^*$ is compact so that we can apply Fredholm theory to conclude $(-I+\mathscr{L}^*)^{-1}$ exists on $(L^q \times L^q)_0(\Omega)$.

THEOREM (3.2.10). *For* $\boldsymbol{\phi} \in (L^q \times L^q)_0(\partial\Omega)$ *there exists a function* v *such that*

$$\Delta^2 v(x) = 0, \qquad X \in \Omega$$

$$\lim_{\substack{X \in Q \\ X \in \Gamma(Q)}} \mathbf{L}_{(\mathbf{n}_Q)} v(X) = \boldsymbol{\phi}(Q) \qquad a.e.$$

Proof. Let $\phi = (-I + \mathscr{L}^*)^{-1}\,\phi$ and set $v = v_m(\phi_0; X)$. Then by Theorem (2.3.5),

$$\begin{aligned}\lim_{\substack{X \in \Gamma \\ X \in \Gamma}} \mathbf{L}_{(n_Q)} v_m(\phi_0; X) &= (-I + \mathscr{L}^*)(\phi_0) \\ &= (-I + \mathscr{L}^*)(-I + \mathscr{L}^*)^{-1}(\phi) \\ &= \phi.\end{aligned}$$

This concludes the paper as Theorem (3.2.10) is a restatement of Theorem (1.2.1).

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